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Boundary jets of holomorphic maps between strongly pseudoconvex domains

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Abstract

In this paper we consider jets taken at a fixed boundary point of germs of holomorphic diffeomorphisms which send one strongly pseudoconvex domain into another. We completely describe possible first and second jets and conditions of extremality in terms of the Chern–Moser normal forms of the domains.

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1. Introduction

The goal of this paper is to initiate a study of holomorphic mappings F between two domains D and D' in \mathbb{C}^{n+1} , sending D to a subset $F(D) \subset D'$ whose shape approximates D' as much as possible. It is known since Poincaré [9] and subsequent work by Tanaka [10], Chern and Moser [6] and Fefferman [7] that in general, there does not exist any biholomorphic maps between two given bounded strongly pseudoconvex domains in \mathbb{C}^{n+1} with $n \geq 1$ (see [2] for a survey of further results in this direction). On the other hand, there are clearly many biholomorphic maps F from D to open subsets of D' . *Can one impose any condition on F making it “close” to be biholomorphic between D and D' without losing the existence of such maps in general?*

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If one does not keep the condition $F(D) \subset D'$, the Chern–Moser theory gives an estimate on the maximal possible contact order between the boundaries of D and $F(D)$. Our goal here is to study this question *under the assumption* $F(D) \subset D'$.

We treat the problem locally at the given points $p \in \partial D$ and $p' \in \partial D'$ and consider the set $\mathcal{J}_{p,p'}^0(D, D')$ of all germs F at p of holomorphic maps from D to D' sending p to p' in the non-tangential sense. That is, an element in $\mathcal{J}_{p,p'}^0(D, D')$ is represented by a holomorphic map $F: U \cap D \rightarrow D'$ with U being a neighborhood of p , such that $F(Z_n) \rightarrow p'$ whenever $Z_n \rightarrow p$ non-tangentially in $U \cap D$ (i.e. the distance from Z_n to p does not exceed its distance to ∂D times a constant multiple). We shall assume both D and D' to be strongly pseudoconvex with smooth boundaries and choose local holomorphic coordinates (z, w) and (z', w') near p and p' , respectively, where $p = p' = 0$ and D, D' are locally given by

$$\operatorname{Im} w > \|z\|^2 + O(3), \quad \operatorname{Im} w' > \|z'\|^2 + O(3), \quad (1.1)$$

where $\|z\| := |z_1|^2 + \dots + |z_n|^2$. In order to speak about a contact order between $F(D)$ and D' we need to introduce a differential of F at the boundary point p , which we again understand in the non-tangential sense (see Section 2 for precise definition).

The first question is what the possible non-tangential differentials that may occur in this way are. As a first preliminary result we give a complete characterization in terms of the singular values. Recall that every complex $n \times n$ matrix C admits its singular value decomposition $C = U_1 D U_2$, where $U_1, U_2 \in \mathbf{U}(n)$ are unitary and D is diagonal with real non-negative entries $\mu_1 \geq \dots \geq \mu_n \geq 0$ (this can be shown, e.g., by using the polar decomposition $C = UH$ with U unitary and H hermitian and by further diagonalizing H). The entries of D are uniquely determined by C . We have the following characterization:

Proposition 1.1. *A linear map $L: \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{n+1}$ is the non-tangential differential of a germ $F \in \mathcal{J}_{p,p'}^0(D, D')$ if and only if, in the chosen coordinates, it is of the form*

$$L = \begin{pmatrix} C & A \\ 0 & \lambda \end{pmatrix}, \quad (1.2)$$

where $\lambda > 0$ is a real number, $A \in \mathbb{C}^n$ is a complex vector and C is a complex $n \times n$ matrix whose singular values $\mu_1 \geq \dots \geq \mu_n \geq 0$ satisfy $\mu_j \leq \sqrt{\lambda}$ for $j = 1, \dots, n$.

In particular from (1.2) it follows that a germ F of holomorphic map from D to D' sending 0 to 0 which is non-tangentially differentiable at 0 is a *contact map*, in the sense that its non-tangential differential maps the complex tangent space $T_0^c \partial D$ into $T_0^c \partial D'$. In case D, D' are bounded strongly pseudoconvex domains and $F: D \rightarrow D'$ is holomorphic (i.e. it is not just a germ near a boundary point), this latter fact follows also from Abate's generalization of the classical Julia–Wolff–Carathéodory theorem [1]. Thus, in a certain sense, Proposition 1.1 can be interpreted as a Julia–Wolff–Carathéodory theorem in the local.

If F is as in Proposition 1.1, set $\alpha_j := \mu_j / \sqrt{\lambda}$ for $j = 1, \dots, n$. We call the numbers $1 \geq \alpha_n \geq \dots \geq \alpha_1 \geq 0$ the *singular values of the (non-tangential) differential of F at p* . It turns out that these numbers do not depend on the choice of coordinates (z, w) and (z', w') provided (1.1) holds (see Lemma 2.1). On the other hand, one can easily eliminate A by composing F with a suitable automorphism of the corresponding Siegel domain $\operatorname{Im} w > \|z\|^2$ of the form

$$g_a(z, w) := \frac{(z + aw, w)}{1 - 2i\langle z, a \rangle - i\|a\|^2 w}. \quad (1.3)$$

Hence these are the only “first order” invariants of F at p and, roughly speaking, they read the ratios of “squeezing” by F in complex tangent directions comparing to the normal direction. The nearer to 1 the singular values are, the more similar to D' the image $F(D)$ looks like near p' .

As the next step, we study the conditions on the “higher order jets” of F at p . In order to make it meaningful to talk about jets at boundary points, we shall assume F to have smooth extension through the boundary. That is, we consider the subset $\mathcal{J}_{p,p'}(D, D')$ of $\mathcal{J}_{p,p'}^0(D, D')$ consisting of all germs F having representatives extending smoothly to some neighborhoods of p . It is not hard to see that if $\alpha_j < 1$ for all j , then there are no restrictions on the set of possible higher order jets of maps in $\mathcal{J}_{p,p'}(D, D')$ whose differentials at p have the given singular values $1 > \alpha_1 \geq \dots \geq \alpha_n \geq 0$. However, our Proposition 1.1 above implies that even the choice of F with $1 = \alpha_1 = \dots = \alpha_n$ is always possible giving a better contact of $F(D)$ with D' . Our next question is now to examine the possible restrictions on the higher order jets of F in this “extreme case.”

Let $F \in \mathcal{J}_{p,p'}(D, D')$ be such that all the singular values of its differential at p are 1. Then in view of the remarks above, one can choose the coordinates (z, w) and (z', w') preserving (1.1) such that dF_p becomes the identity id . The property of having $dF_p = \text{id}$ in suitable coordinates where (1.1) holds, admits a natural higher order generalization:

Definition 1.2. A germ $F \in \mathcal{J}_{p,p'}(D, D')$ is said to be k -flat if there exist local coordinates (z, w) and (z', w') vanishing at p and p' , respectively, where the hypersurfaces ∂D and $\partial D'$ are respectively in their Chern–Moser normal forms and such that $F = \text{id} + o(k)$.

In other words, a germ $F \in \mathcal{J}_{p,p'}(D, D')$ is k -flat if and only if there exist two local biholomorphic maps of \mathbb{C}^{n+1} , h at p and g at p' , such that $h(p) = g(p') = 0$, $h(\partial D)$ and $g(\partial D')$ are in their Chern–Moser normal forms and $g \circ F \circ h^{-1} = \text{id} + o(k)$, where $H = o(k)$ means that H and all its derivatives of order less than $k + 1$ are 0 at 0. Thus a germ $F \in \mathcal{J}_{p,p'}(D, D')$ is 1-flat if and only if all singular values of dF_p are 1 and, as a consequence of Proposition 1.1, there always exist 1-flat germ for any D and D' .

Remark 1.3. It follows from the construction of the normal form in [6] that the Chern–Moser normalizations in Definition 1.2 are only needed to be chosen for the terms of weight $\leq k$, where as usual, the weight of z and \bar{z} is 1 and of w is 2.

Using Chern–Moser normal forms, we give a complete description of the second order jets for maps in $\mathcal{J}_{p,p'}(D, D')$ whose first jet is the identity (Theorem 3.1). In particular it turns out that the space of possible second order jets has its interior described by simple algebraic inequalities. That is, for any 2-jet in the interior, there exists a germ $F \in \mathcal{J}_{p,p'}(D, D')$ with that jet and no further restrictions arise on the possible jets of order three or higher. This gives a more precise description of possible 1-flat germs.

In contrast to 1-flat germs, 2-flat germs may not exist at all for some D and D' . This latter fact is somewhat related to the rigidity phenomena for *self-maps* known as “Burns–Krantz type theorems” (see [3,5,8]). We show that the existence of 2-flat germs implies a nontrivial geometric condition on D and D' expressed as follows. We say that two real hypersurfaces M and M' in \mathbb{C}^{n+1} passing through a point q are *tangent at q up to weighted order k* if, for some (and

hence any) local defining function ρ of M' and some (and hence any) local parametrization $\gamma: \mathbb{C}^n \times \mathbb{R} \rightarrow \mathbb{C}^{n+1}$ of M with $\gamma(0) = q$ and $d\gamma_0(\mathbb{C}^n \times \{0\})$ being the complex tangent space of M at q , the composition $\rho \circ \gamma$ vanishes at 0 up to weighted order k , where as before we assign weight 1 to the coordinates in $z \in \mathbb{C}^n$ and weight 2 to the coordinate in $u \in \mathbb{R}$. We now call $(\partial D, p)$ and $(\partial D', p')$ *equivalent up to weighted order k* if there exists a local holomorphic diffeomorphism of \mathbb{C}^{n+1} near p , sending p to p' and ∂D to another real hypersurface, which is tangent to $\partial D'$ up to weighted order k at p' .

Our result for 2-flat germs can now be stated as follows.

Theorem 1.4. *Let $D, D' \subset \mathbb{C}^{n+1}$ be two domains with smooth boundaries such that $p \in \partial D$, $p' \in \partial D'$ and $\partial D, \partial D'$ are strongly pseudoconvex at p and p' , respectively. Then there exist 2-flat maps in $\mathcal{J}_{p,p'}(D, D')$ if and only if $(\partial D, p)$ and $(\partial D', p')$ are equivalent up to weighted order 5.*

The outline of the paper is the following. In Section 2 we prove the “only if” part of Proposition 1.1 and discuss the first jets. In Section 3 we recall the Chern–Moser theory as needed for our purposes, describe the possible second jets for 1-flat germs (Theorem 3.1), finish the proof of Proposition 1.1, give some equivalent conditions for 2-flatness and prove Theorem 1.4. Finally, there is an Appendix A where we collected some auxiliary results needed in the various proofs.

2. First order jets

Let $F: D \rightarrow \mathbb{C}^m$ be holomorphic for some m . We say that F is *non-tangentially differentiable* at p if there exists a point $p' \in \mathbb{C}^m$ and a linear map $dF_p: \mathbb{C}^{n+1} \rightarrow \mathbb{C}^m$ such that

$$F(Z_k) = p' + dF_p(Z_k - p) + o(\|Z_k - p\|), \quad k \rightarrow \infty, \quad (2.1)$$

holds for any sequence (Z_k) of points in D converging non-tangentially to p . We call dF_p the *non-tangential differential* of F at p .

We shall consider the case when D, D' are domains in \mathbb{C}^{n+1} with smooth boundaries and strongly pseudoconvex points $p \in \partial D$, $p' \in \partial D'$. In the sequel, we shall assume $p = p' = 0$ and choose local holomorphic coordinates $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ and $(z', w') \in \mathbb{C}^n \times \mathbb{C}$ vanishing at the origin such that D and D' are locally given by

$$\begin{aligned} D &= \{\operatorname{Im} w > \|z\|^2 + O(|(z, \operatorname{Re} w)|^3)\}, \\ D' &= \{\operatorname{Im} w' > \|z'\|^2 + O(|(z', \operatorname{Re} w')|^3)\}. \end{aligned} \quad (2.2)$$

We will obtain the “only if” statement of Proposition 1.1 as consequence of the following lemma.

Lemma 2.1. *Let D, D' be two domains in \mathbb{C}^{n+1} of the form (2.2) and $F \in \mathcal{J}_{0,0}^0(D, D')$ be non-tangentially differentiable at 0 with differential dF_0 . Then dF_0 is given by the block matrix*

$$dF_0 = \begin{pmatrix} C & A \\ 0 & \lambda \end{pmatrix}, \quad (2.3)$$

where $\lambda > 0$ is a real number, $A \in \mathbb{C}^n$ is a complex vector and C is a complex $(n \times n)$ -matrix whose singular values $\mu_1 \geq \dots \geq \mu_n \geq 0$ satisfy $\mu_j \leq \sqrt{\lambda}$ for $j = 1, \dots, n$. The ratios $\mu_j/\sqrt{\lambda}$ for $j = 1, \dots, n$ are invariant under coordinates changes preserving (2.2).

For the proof we need the following elementary result, whose proof is supplied for the reader's convenience.

Lemma 2.2. *Let D be a domain in \mathbb{C}^{n+1} having 0 as a smooth boundary point and $F: D \rightarrow \mathbb{C}^m$ be holomorphic. If dF_0 is the non-tangential differential of F at 0, then $dF_{Z_k} \rightarrow dF_0$ for any sequence (Z_k) in D converging non-tangentially to 0.*

Proof. Without loss of generality, $F(0) = 0$ and $dF_0 = 0$. Let $\{Z_k\}$ be any sequence in D converging non-tangentially to $0 \in \partial D$. It suffice to show that $\frac{\partial F}{\partial Z^l}(Z_k) \rightarrow 0$ for every $l = 1, \dots, n+1$, where we use the notation $Z = (Z^1, \dots, Z^{n+1})$. We give a proof for $l = 1$, the other cases being completely analogous. Since $\{Z_k\}$ converges non-tangentially in D , there exists $\varepsilon > 0$ such that, for any other sequence $\{\tilde{Z}_k\}$ with $\|\tilde{Z}_k - Z_k\| \leq \varepsilon \|Z_k\|$, one has $\tilde{Z}_k \in D$ for all k and $\{\tilde{Z}_k\}$ also converges to 0 non-tangentially in D . By the Cauchy integral formula, we have

$$\begin{aligned} \left| \frac{\partial F}{\partial Z^1}(Z_k) \right| &= \left| \frac{1}{2\pi i} \int_{|\zeta - Z_k^1| = \varepsilon \|Z_k\|} \frac{F(\zeta, Z_k^2, \dots, Z_k^{n+1})}{(\zeta - Z_k^1)^2} d\zeta \right| \\ &\leq \frac{1}{\varepsilon \|Z_k\|} \max_{\zeta} |F(\zeta, Z_k^2, \dots, Z_k^{n+1})|. \end{aligned} \quad (2.4)$$

It remains to choose ζ with $|\zeta - Z_k^1| = \varepsilon \|Z_k\|$ such that the maximum in (2.4) is attained for $\tilde{Z}_k := (\zeta, Z_k^2, \dots, Z_k^{n+1})$ and use (2.1) with Z_k replaced by \tilde{Z}_k . \square

Proof of Lemma 2.1. We first observe that dF_0 must send the upper half-space $\operatorname{Im} w \geq 0$ into itself. Otherwise there would exist a non-tangentially convergent sequence $Z_k = v\varepsilon_k$, where $v \in \{(z, w): \operatorname{Im} w > 0\}$ is a vector with $dF_0(v)$ not contained in $\operatorname{Im} w \geq 0$ and $\{\varepsilon_k\}$ a sequence of positive real numbers converging to 0. The latter would be in contradiction with (2.1) and (2.2). Hence dF_0 sends the real hyperplane $\operatorname{Im} w = 0$ into itself and, since it is complex-linear in view of Lemma 2.2, also the complex hyperplane $w = 0$ into itself. Putting everything together, we conclude that dF_0 is of the form (2.3) with some matrices C and A and a real number $\lambda \geq 0$.

The second step consists of showing that $\lambda > 0$. Suppose, on the contrary, that $\lambda = 0$. Since D' is strongly pseudoconvex at 0, it is easy to construct a continuous plurisubharmonic (peak) function φ defined in a neighborhood of 0 in \mathbb{C}^{n+1} such that $\varphi(0) = 0$, $d\varphi_0 = -d(\operatorname{Im} w)$ and $\varphi(Z) < 0$ for $Z \in \bar{D}' \setminus \{0\}$. Furthermore, it is easy to extend φ to a continuous plurisubharmonic function ψ defined on the whole D' by setting

$$\psi(Z) := \begin{cases} \max(\varphi(Z), -\varepsilon) & \text{for } \|Z\| < \delta, \\ -\varepsilon & \text{otherwise,} \end{cases} \quad (2.5)$$

where $\delta > 0$ and $\varepsilon > 0$ are chosen such that $\varphi(Z) < -\varepsilon$ for $Z \in D'$ with $\|Z\| = \delta$. Note that ψ coincides with φ in a neighborhood of 0 in \bar{D}' . Then $\lambda \neq 0$ follows from the Hopf lemma applied

to $\psi \circ F$ restricted to a disk in the complex line $\{(z, w) : z = 0\}$ that is contained in D and tangent to the boundary ∂D at 0.

The third step is to show that the ratios $\mu_j/\sqrt{\lambda}$ do not depend on the coordinates chosen and that the inequalities $\mu_j/\sqrt{\lambda} \leq 1$ hold. Using the singular value decomposition of C , we can compose F with suitable unitary linear transformations of \mathbb{C}^{n+1} on the right and on the left, such that both forms (2.2) are preserved and C becomes diagonal with real entries $\mu_1 \geq \dots \geq \mu_n \geq 0$ equal to its singular values. Furthermore, composing with a dilation $(z, w) \mapsto (\lambda z, |\lambda|^2 w)$, we may assume that $\lambda = 1$.

Now consider any changes of coordinates $(z, w) \mapsto \varphi_1(z, w)$ and $(z', w') \mapsto \varphi_2(z', w')$ preserving (2.2). Then the differentials $(d\varphi_1)_0$ and $(d\varphi_2)_0$ must be of the form

$$\begin{pmatrix} U_j \lambda_j & * \\ 0 & \lambda_j^2 \end{pmatrix}, \quad j = 1, 2,$$

where λ_j 's are real positive and U_j 's are unitary. Furthermore, in order to keep the above normalization of dF_0 , we must have $\lambda_1 = \lambda_2$. Then in these new coordinates, we have

$$dF_0 = \begin{pmatrix} U_2 C U_1^{-1} & * \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \tilde{C} & * \\ 0 & 1 \end{pmatrix}, \quad (2.6)$$

and it follows that the singular values of \tilde{C} coincide with those of C . This shows that the ratios $\mu_j/\sqrt{\lambda}$ are invariants of dF_0 .

To show that $\mu_j/\sqrt{\lambda} \leq 1$ for all $j = 1, \dots, n$, or $\mu_j \leq 1$ in our normalization, it suffices to show that $\|C\| \leq 1$. By contradiction, suppose that $\|C\xi\| > 1$ for a vector $\xi \in \mathbb{C}^n$ with $\|\xi\| = 1$. We change local holomorphic coordinates in \mathbb{C}^{n+1} near 0 such that ∂D and $\partial D'$ are approximated by the ball $\{\|Z - (0, i/2)\| < 1\}$ up to order 3 at 0. Such coordinate change can be chosen to be the identity up to order 2, so that the matrix of dF_0 does not change. Then, in view of Lemma 2.2, we can choose discs

$$f_k: \Delta \rightarrow D, \quad f_k(\zeta) := \left(\zeta \left(1 - \frac{\varepsilon}{2} \right) \frac{\xi}{k}, \frac{i}{k^2} \right) \in \mathbb{C}^n \times \mathbb{C}, \quad (2.7)$$

where Δ is the unit disc in \mathbb{C} , with $\varepsilon > 0$ sufficiently small such that

$$\left\| \frac{1-\varepsilon}{1+\varepsilon} dF_{Z_k}^z(\xi) \right\| > 1, \quad (2.8)$$

where $F^z \in \mathbb{C}^n$ denotes the tangential component of F and $Z_k := f_k(0)$. By the attraction property (Lemma A.1), for $\eta := \frac{1-\varepsilon}{1+\varepsilon/2}$, we may assume that the images $F(f_k(\eta\Delta))$ are contained in a sufficiently small neighborhood of 0. Choose $r_k > 0$ such that the central projection π_k from $(0, -r_k)$ onto the hyperplane $\{w = i/k^2\}$ sends the ball $\{\|Z - (0, i/2)\| < 1\}$ in \mathbb{C}^{n+1} into the ball with center $(0, i/k^2)$ and radius $\frac{1+\varepsilon/2}{k}$ in the hyperplane $\{w = i/k^2\}$. Then there exists sufficiently small neighborhood U of 0 such that, for k sufficiently large, π_k sends $U \cap D'$ into the ball with center $(0, i/k^2)$ and radius $(1+\varepsilon)/k$. Together with (2.8), we reach a contradiction with the Schwarz lemma for $\pi_k \circ F \circ f_k$ restricted to $\eta\Delta$. \square

Remark 2.3. We say that a holomorphic map F from D into D' which extends smoothly to p and maps p to q and sends ∂D into $\partial D'$ up to order k at p if, for some (and hence any) local

defining function ρ_2 of $\partial D'$, its pullback $\rho_2 \circ F$ is $o(k)$ on ∂D (e.g., F sends ∂D into $\partial D'$ up to order 1 if $dF_p(T_p \partial D) \subseteq T_q \partial D'$). From the previous discussion it is clear that if F extends smoothly to p and sends ∂D into $\partial D'$ up to order 2, then the singular values of dF_p at p are all equal to 1.

Let now $F \in \mathcal{J}_{0,0}^0(D, D')$. In the proof of Lemma 2.1 we have seen that in case all singular values of the differential of F at 0 are equal to 1, we can choose coordinates (z, w) and (z', w') such that (2.2) holds and

$$dF_0 = \begin{pmatrix} \text{id} & A \\ 0 & 1 \end{pmatrix},$$

for some complex vector $A \in \mathbb{C}^n$. Using automorphisms g_A of the Siegel domain $\{(z, w) \in \mathbb{C}^n \times \mathbb{C} : \text{Im } w > \|z\|^2\}$ given by (1.3) we can replace F by $g_{-A} \circ F$ to make its differential at the origin equal to the identity. Here and in the sequel we set

$$\langle z, \zeta \rangle := z_1 \bar{\zeta}_1 + \cdots + z_n \bar{\zeta}_n. \quad (2.9)$$

Note that g_A preserves the form (2.2).

According to [6], we can find germs of biholomorphisms h_1 and h_2 such that $h_1(0) = h_2(0) = 0$ and $d(h_1)_0 = d(h_2)_0 = \text{id}$ and $h_1(D), h_2(D')$ are in their Chern–Moser normal forms. If $dF_0 = \text{id}$, we have $h_2 \circ F \circ h_1^{-1} = \text{id} + o(1)$. Therefore we have:

Corollary 2.4. *A germ $F \in \mathcal{J}_{p,p'}^0(D, D')$ is 1-flat if and only if all singular values of its differential at p are equal to 1.*

3. Second order jets for 1-flat maps

As a matter of notations, for $m \in \mathbb{N}$, we use the symbol $O(m)$ to represent any (smooth) function which vanishes at the origin together with its derivatives up to order less than m . The symbol $o(m)$ for $m \in \mathbb{N}$ means that also the m th derivative is zero at the origin. Whenever we need to state explicitly that a function depending on several (complex or real) variables vanishes at the origin together with all its partial derivatives with respect to a certain variable—say u —up to the order m , we write such a function as $O(u^m)$. Also we freely mix and add these notations. For instance the function $3u^4v^3 + v^2u^5$ can be written as $O(7)$, or as $O(v^2)$ or $O(v^2) + O(u^4)$ or even as $O(u^4) + O(5)$. The same notation is used for the small Landau's symbol o .

In this section we assume $F \in \mathcal{J}_{p,p'}(D, D')$ to be 1-flat. Recall that our notation $\mathcal{J}_{p,p'}(D, D')$ was reserved for the holomorphic map germs having smooth extensions to some neighborhoods of p . Arguing as in the previous section we may assume $p = p' = 0$, $dF_0 = \text{id}$ and $\partial D, \partial D'$ given (locally) by expressions of the form $\text{Im } w = \|z\|^2 + O(3)$. In order to simplify the notation, we use the symbol $\mathcal{J}(D, D')$ to denote the germs of holomorphic maps from D to D' which are smooth at 0 and such that $F(0) = 0$.

As a matter of notation, if $f: \mathbb{C}^{n+1} = \mathbb{C}^n \times \mathbb{C} \rightarrow \mathbb{C}^{n+1}$ is expandable at the origin, with homogeneous expansion $f(z, w) = \sum_v f_v(z, w)$, we are going to denote by $f_j^z(z, w)$ the projection to \mathbb{C}_z^n of the homogeneous polynomial vector f_j and by f_j^w the projection to \mathbb{C}_w . Moreover, for a homogeneous polynomial $P(z, w)$ of degree j , we write $P(z, w) = \sum_{v=0}^j P_{v,j-v}(z, w)$, for $P_{l,k}(z, w) = C_{lk}(z)w^k$, where $C_{lk}(z)$ is a homogeneous polynomial of degree l in the z 's.

To deal with jets of order two we need however to have better expansions for the normal forms of the domains.

Following [6], we assign weight 1 to z_j, \bar{z}_j (for $j = 1, \dots, n$) and weight 2 to $u = \operatorname{Re} w$. A real polynomial $P(z, \bar{z}, u)$ is of *weighted degree* m if it is a linear combination of monomials of type $z_{j_1} \cdots z_{j_k} u^l$ with $k + 2l = m$. With this notation, Chern–Moser normal forms for ∂D and $\partial D'$ can be written as

$$\begin{aligned}\partial D &= \left\{ \operatorname{Im} w = \|z\|^2 + \sum_{\mu \geq 4} \varphi_\mu(z, \bar{z}, \operatorname{Re} w) \right\}, \\ \partial D' &= \left\{ \operatorname{Im} w = \|z\|^2 + \sum_{\mu \geq 4} \varphi'_\mu(z, \bar{z}, \operatorname{Re} w) \right\},\end{aligned}\quad (3.1)$$

where φ_μ and φ'_μ are real weighted homogeneous polynomials of weighted degree μ which are linear combinations of monomials, each of which is divisible by $z_{j_1} z_{j_2} \bar{z}_{k_1} \bar{z}_{k_2}$ for some $j_1, j_2, k_1, k_2 \in \{1, \dots, n\}$. In particular, $\varphi_4, \varphi'_4, \varphi_5, \varphi'_5$ have no dependence in $\operatorname{Re} w$. Also, $\operatorname{tr}(\varphi_4) \equiv 0$ and $\operatorname{tr}^2(\varphi_5) \equiv 0$ and similarly for φ'_4, φ'_5 , where, if $Q_{(j,k)}(z, \bar{z})$ is a polynomial of degree j in z_α and degree k in \bar{z}_β given by

$$Q_{(j,k)}(z, \bar{z}) = \sum a_{\alpha_1 \dots \alpha_j \beta_1 \dots \beta_k} z_{\alpha_1} \cdots z_{\alpha_j} \bar{z}_{\beta_1} \cdots \bar{z}_{\beta_k},$$

with the $a_{\alpha_1 \dots \alpha_j \beta_1 \dots \beta_k}$'s symmetric with respect to $\alpha_1, \dots, \alpha_j$ and with respect to β_1, \dots, β_k , then

$$\operatorname{tr}(Q_{(j,k)}) := \sum_{\alpha_1, \dots, \alpha_{j-1}, \beta_1, \dots, \beta_{k-1}} \left(\sum_{\alpha_j = \beta_k} a_{\alpha_1 \dots \alpha_j \beta_1 \dots \beta_k} \right) z_{\alpha_1} \cdots z_{\alpha_{j-1}} \bar{z}_{\beta_1} \cdots \bar{z}_{\beta_{k-1}}.$$

Actually a Chern–Moser normal form as defined in [6] involves further trace conditions on higher order terms that we will not need here. Notice that the Chern–Moser normal form of a domain is not unique, but it is parametrized by the automorphisms of the quadric $\{\operatorname{Im} w = \|z\|^2\}$ fixing the origin.

Theorem 3.1. *Let D and D' be in their Chern–Moser normal forms (3.1) and $F \in \mathcal{J}_{0,0}(D, D')$ with $dF_0 = \operatorname{id}$. Then*

$$F(z, w) = (z, w) + (F_{1,1}^z(z, w) + F_{0,2}^z(w), F_{0,2}^w(w)) + O(3) \quad (3.2)$$

with

$$\begin{aligned}\operatorname{Im} F_{0,2}^w(1) &\geq 0, \\ [\operatorname{Re}\langle z, F_{1,1}^z(z, 1) \rangle - \|z\|^2 \operatorname{Re} F_{0,2}^w(1)]^2 \\ &\leq \operatorname{Im} F_{0,2}^w(1) [\varphi_4(z) - \varphi'_4(z) - 2\|z\|^2 \operatorname{Im}\langle z, F_{1,1}^z(z, 1) \rangle - \|z\|^4 \operatorname{Im} F_{0,2}^w(1)]\end{aligned}\quad (3.3)$$

and one has $F_{3,0}^w \equiv 0$ and

$$\varphi_4(z) - \varphi'_4(z) - 2\|z\|^2 \operatorname{Im}\langle z, F_{1,1}^z(z, 1) \rangle \geq 0. \quad (3.4)$$

On the other hand, for any choice of the 2nd order terms in (3.2) satisfying (3.3) with strict inequalities for all $z \in \mathbb{C}^n \setminus \{0\}$ there exists $F \in \mathcal{J}_{0,0}(D, D')$ of the form (3.2).

Proof. We shall use Corollary A.6 applied to the basic condition $F(D) \subseteq D'$. In view of (3.1), a parametrization for ∂D is given by

$$\mathbb{C}^n \times \mathbb{R} \ni (z, u) \mapsto Z = \left(z, u + i\|z\|^2 + i \sum_{\mu \geq 4} \varphi_\mu^1(z, \bar{z}, u) \right). \quad (3.5)$$

Therefore the basic condition becomes

$$\begin{aligned} & \sum_{\mu \geq 4} \varphi_\mu(z, \bar{z}, u) + \sum_{k \geq 2} \operatorname{Im} F_k^w(Z) \\ & \geq 2 \sum_{k \geq 2} \operatorname{Re} \langle z, F_k^z(Z) \rangle + \left\| \sum_{k \geq 2} F_k^z(Z) \right\|^2 + \sum_{\mu \geq 4} \varphi'_\mu(F^z(Z), \overline{F^z(Z)}, \operatorname{Re} F^w(Z)), \end{aligned} \quad (3.6)$$

where Z is as in (3.5) and F_k denotes the component of weight k . Expanding (3.6) up to weighted order two and applying Corollary A.6 we have

$$\operatorname{Im} F_{2,0}^w \geq 0. \quad (3.7)$$

Since $z \mapsto F_{2,0}^w(z)$ is holomorphic this means that $F_{2,0}^w \equiv 0$.

Now expanding (3.6) up to weighted order three and applying Corollary A.6 yields

$$\operatorname{Im} F_{3,0}^w(z) + \operatorname{Im} F_{1,1}^w(z, u + i\|z\|^2) \geq 2 \operatorname{Re} \langle z, F_{2,0}^z(z) \rangle. \quad (3.8)$$

Separating into terms of different bi-degree types and again using Corollary A.6 we obtain two inequalities, namely

$$\operatorname{Im} F_{1,1}^w(z, u) \geq 0 \quad (3.9)$$

and

$$\operatorname{Im} F_{3,0}^w(z) + \operatorname{Im} F_{1,1}^w(z, i\|z\|^2) \geq 2 \operatorname{Re} \langle z, F_{2,0}^z(z) \rangle. \quad (3.10)$$

Inequality (3.9) is indeed an equality because $F_{1,1}^w(z, u)$ is linear in u and, since $z \mapsto F_{1,1}^w(z, u)$ is holomorphic for any fixed u , it follows furthermore that $F_{1,1}^w \equiv 0$. Now applying Lemma A.8 to (3.10) we obtain $\operatorname{Im} F_{3,0}^w \equiv 0$ and hence $F_{3,0}^w \equiv 0$ for $z \mapsto F_{3,0}^w(z)$ is holomorphic, and, consequently, $\operatorname{Re} \langle z, F_{2,0}^z(z) \rangle \equiv 0$ for all $z \in \mathbb{C}^n$. This last equality clearly implies $F_{2,0}^z \equiv 0$.

Therefore 1-flatness implies that all terms of weighted order two and three in the expansion of (3.6) are zero. Now we pass to the weighted order four:

$$\begin{aligned} & \varphi_4(z, \bar{z}) + \operatorname{Im} F_{4,0}^w(z) + \operatorname{Im} F_{2,1}^w(z, u + i\|z\|^2) + \operatorname{Im} F_{0,2}^w(u + i\|z\|^2) \\ & \geq 2 \operatorname{Re} \langle z, F_{3,0}^z(z) \rangle + 2 \operatorname{Re} \langle z, F_{1,1}^z(z, u + i\|z\|^2) \rangle + \varphi'_4(z, \bar{z}). \end{aligned} \quad (3.11)$$

By Corollary A.6, looking at terms of the lowest degree in (z, \bar{z}) we obtain that $\operatorname{Im} F_{0,2}^w(u) \geq 0$ and, since the dependence on u is quadratic, this is equivalent to $\operatorname{Im} F_{0,2}^w(1) \geq 0$.

Now we can set $u = t\|z\|^2$ with $t \in \mathbb{R}$ in (3.11), and apply Remark A.7 to terms of bi-degree $(2, 2)$ in (z, \bar{z}) :

$$t^2\|z\|^4 \operatorname{Im} F_{0,2}^w(1) + 2t(\|z\|^4 \operatorname{Re} F_{0,2}^w(1) - \|z\|^2 \operatorname{Re}\langle z, F_{1,1}^z(z, 1) \rangle) \\ + \varphi_4(z, \bar{z}) - \varphi_4'(z, \bar{z}) - 2\|z\|^2 \operatorname{Im}\langle z, F_{1,1}^z(z, 1) \rangle - \|z\|^4 \operatorname{Im} F_{0,2}^w(1) \geq 0. \quad (3.12)$$

For $z \neq 0$ fixed, the left-hand side of (3.12) must be greater than or equal 0 for all t , which is equivalent to (3.3) and (3.4) (for $\operatorname{Im} F_{0,2}^w(1) \neq 0$, (3.4) follows from (3.3)).

Finally, if both inequalities in (3.3) are strict for any $z \neq 0$, then the lowest weighted order nontrivial homogeneous term in (3.6) is positive for $(z, u) \neq 0$ if we choose F to be of the form (3.2) without higher order terms. Therefore (3.6) will always hold in a neighborhood of the origin. This proves the last statement. \square

Remark 3.2. It is apparent from the proof that the conclusions of Theorem 3.1 still hold with D, D' being given in Chern–Moser normal forms only up to weighted order 5 at 0.

Now we are in the position to end the proof of Proposition 1.1.

End of the proof of Proposition 1.1. In order to complete the proof of Proposition 1.1 we need to show that given a matrix L as in (1.2), there exists $F \in \mathcal{J}_{0,0}(D, D')$ such that $dF_0 = L$. Using transformations tangent to id we can suppose that D, D' are in their Chern–Moser normal form (at least up to weighted order four). Finally, acting with an automorphism (1.3) and a dilation $(z, w) \mapsto (\lambda z, |\lambda|^2 w)$ on the left and with unitary transformations $(z, w) \mapsto (Uz, w)$ on both sides, we can reduce the general case to that of

$$L = \begin{pmatrix} \Delta & 0 \\ 0 & 1 \end{pmatrix},$$

where Δ is a diagonal matrix with entries the singular values α_j 's, with $0 \leq \alpha_n \leq \dots \leq \alpha_1 \leq 1$. Now the argument is similar to that in the proof of Theorem 3.1. We look for F of the form $F(z, w) = (\Delta z, w) + \sum_{k \geq 2} (F_k^z(z, w), F_k^w(z, w))$ and impose the condition $F(\partial D) \subseteq \partial D'$. Parametrizing ∂D with (3.5) we obtain

$$\|z\|^2 + \sum_{\mu \geq 4} \varphi_\mu(z, \bar{z}, w) + \sum_{k \geq 2} \operatorname{Im} F_k^w(Z) \\ \geq \|\Delta z\|^2 + 2 \sum_{k \geq 2} \operatorname{Re}\langle \Delta z, F_k^z(Z) \rangle + \left\| \sum_{k \geq 2} F_k^z(Z) \right\|^2 + \sum_{\mu \geq 4} \varphi'_\mu(F^z(Z), \overline{F^z(Z)}, \operatorname{Re} F^w(Z)).$$

If all entries in Δ are < 1 , we choose $F_{0,2}^w$ with $\operatorname{Im} F_{0,2}^w(1) > 0$ and $F_k^z = F_{k+1}^w = 0$ for $k \geq 2$, then it follows that $F \in \mathcal{J}(D, D')$, that $F(\partial D) \subset \partial D'$ near 0 and that F has the required differential at 0.

Let $l \leq n$ and suppose that $\alpha_1 = \dots = \alpha_l = 1$ and $\alpha_k < 1$ for $k > l$. Let us write $z = (z', z'') \in \mathbb{C}^l \times \mathbb{C}^{n-l}$. Also, with obvious meaning, write $F_k^z = (F_k^{z'}, F_k^{z''})$. Set $F_k^{z''} = 0$ for $k \geq 0$. The last

statement in Theorem 3.1 gives sufficient conditions for the map $(z', w) \mapsto F(z', 0, w)$ to send the domain $D \cap \{z'' = 0\}$ into the domain $D' \cap \{z'' = 0\}$. Then the appropriate choice of $F_k^{z'}$ together with $\operatorname{Im} F_{0,2}^w(1) > 0$ and the inequality $\|(0, z'')\|^2 > \|\Delta(0, z'')\|^2$ for $z'' \neq 0$ guarantees that F sends ∂D into \bar{D}' near 0. \square

Lemma 3.3. *Let $D, D' \subset \mathbb{C}^n$ be in Chern–Moser normal forms and $F \in \mathcal{J}_{0,0}(D, D')$ be with $dF_0 = \operatorname{id}$ and $\operatorname{Im}\langle z, F_{1,1}^z(z, 1) \rangle \equiv 0$. Then*

$$\varphi_4(z, \bar{z}) \equiv \varphi_4'(z, \bar{z}), \quad \varphi_5(z, \bar{z}) \equiv \varphi_5'(z, \bar{z}) \quad (3.13)$$

and

$$\begin{aligned} F(z, w) = & (z, w) + (F_{1,1}^z(z, w), F_{0,2}^w(w)) \\ & + (F_{1,2}^z(z, w) + F_{0,3}^z(w), F_{0,3}^w(w)) + O(4), \end{aligned} \quad (3.14)$$

where

$$\operatorname{Im} F_{0,2}^w(1) = \operatorname{Im} F_{0,3}^w(1) = 0, \quad \|z\|^2 \operatorname{Re} F_{0,2}^w(1) \equiv \operatorname{Re}\langle z, F_{1,1}^z(z, 1) \rangle \quad (3.15)$$

and, for any z ,

$$BC \geq A^2, \quad B \geq 0, \quad C \geq 0, \quad (3.16)$$

where

$$\begin{aligned} A := & -4\|z\|^4 \operatorname{Im}\langle z, F_{1,2}^z(z, 1) \rangle + \|z\|^2 \varphi_{2,2,1}(z, \bar{z}, 1) - \|z\|^2 \varphi_{2,2,1}'(z, \bar{z}, 1), \\ B := & 3\|z\|^6 \operatorname{Re} F_{0,3}^w(1) - 2\|z\|^4 \operatorname{Re}\langle z, F_{1,2}^z(z, 1) \rangle - \|z\|^4 \|F_{1,1}^z(z, 1)\|^2, \\ C := & -\|z\|^6 \operatorname{Re} F_{0,3}^w(1) + 2\|z\|^4 \operatorname{Re}\langle z, F_{1,2}^z(z, 1) \rangle - \|z\|^4 \|F_{1,1}^z(z, 1)\|^2 \\ & + \varphi_{3,3,0}(z, \bar{z}) - \varphi_{3,3,0}'(z, \bar{z}). \end{aligned} \quad (3.17)$$

Moreover $F_{4,0}^z \equiv 0$, $F_{5,0}^w \equiv 0$, $F_{3,1}^w \equiv 0$.

Proof. We first prove that $\operatorname{Im} F_{0,2}^w(1) = 0$. Indeed, we have $\operatorname{Im} F_{0,2}^w(1) \geq 0$ by Theorem 3.1. If we had $\operatorname{Im} F_{0,2}^w(1) > 0$, then dividing both sides of (3.3) by $\operatorname{Im} F_{0,2}^w(1)$ we would obtain that

$$\varphi_4^1(z) - \varphi_4^2(z) - 2\|z\|^2 \operatorname{Im}\langle z, F_{1,1}^z(z, 1) \rangle - \|z\|^4 \operatorname{Im} F_{0,2}^w(1) \geq 0, \quad (3.18)$$

and therefore $\varphi_4^1(z) - \varphi_4^2(z) \geq 0$ (since $\operatorname{Im}\langle z, F_{1,1}^z(z, 1) \rangle \equiv 0$). Since $\operatorname{tr}(\varphi_4^1(z) - \varphi_4^2(z)) = 0$, the function $\varphi_4^1(z) - \varphi_4^2(z)$ is harmonic and hence would be identically zero by the maximum principle. Then (3.18) would imply $\operatorname{Im} F_{0,2}^w(1) = 0$, a contradiction. Therefore $\operatorname{Im} F_{0,2}^w(1) = 0$ and also $\varphi_4^1(z) - \varphi_4^2(z) \equiv 0$. Hence (3.3) implies $\|z\|^2 \operatorname{Re} F_{0,2}^w(1) \equiv \operatorname{Re}\langle z, F_{1,1}^z(z, 1) \rangle$. This proves (3.15), except for $F_{0,3}^w$.

Summarizing, we have the equality in (3.12) for all z and t . Hence, by Lemma A.8, we must have the equality also in (3.11). In particular, separating types, we obtain the vanishing of

$$F_{4,0}^w, \quad F_{2,1}^w(z, 1), \quad F_{3,0}^z(z), \quad \|z\|^2 \operatorname{Re} F_{2,1}^w(z, 1) - 2 \operatorname{Re} \langle z, F_{3,0}^z(z) \rangle. \quad (3.19)$$

Hence in (3.6) all terms of weighted order less or equal to 4 cancel each other, and we obtain the following inequality for the terms of weighted order 5:

$$\begin{aligned} & \varphi_5(z, \bar{z}) + \operatorname{Im} F_{5,0}^w(z) + \operatorname{Im} F_{3,1}^w(z, w) + \operatorname{Im} F_{1,2}^w(z, w) \\ & \geq 2 \operatorname{Re} \langle z, F_{4,0}^z(z) \rangle + 2 \operatorname{Re} \langle z, F_{2,1}^z(z, w) \rangle + 2 \operatorname{Re} \langle z, F_{0,2}^z(w) \rangle + \varphi'_5(z, \bar{z}), \end{aligned} \quad (3.20)$$

where the terms are evaluated at $(z, w) = (z, u + i\|z\|^2)$. By Corollary A.6, we can pass to the reduced inequality involving only terms of degree 0 in u . These are homogeneous polynomials of the odd degree 5 in (z, \bar{z}) and hence we have the equality

$$\begin{aligned} & \varphi_5(z, \bar{z}) + \operatorname{Im} F_{5,0}^w(z) + \operatorname{Im} F_{3,1}^w(z, i\|z\|^2) + \operatorname{Im} F_{1,2}^w(z, i\|z\|^2) \\ & = 2 \operatorname{Re} \langle z, F_{4,0}^z(z) \rangle + 2 \operatorname{Re} \langle z, F_{2,1}^z(z, i\|z\|^2) \rangle + 2 \operatorname{Re} \langle z, F_{0,2}^z(i\|z\|^2) \rangle + \varphi'_5(z, \bar{z}). \end{aligned} \quad (3.21)$$

Separating types we obtain

$$\operatorname{Im} F_{5,0}^w(z) = 0, \quad (3.22)$$

$$\operatorname{Im} F_{3,1}^w(z, i\|z\|^2) = 2 \operatorname{Re} \langle z, F_{4,0}^z(z) \rangle, \quad (3.23)$$

$$\begin{aligned} \varphi_5(z, \bar{z}) + \operatorname{Im} F_{1,2}^w(z, i\|z\|^2) &= 2 \operatorname{Re} \langle z, F_{0,2}^z(i\|z\|^2) \rangle + 2 \operatorname{Re} \langle z, F_{2,1}^z(z, i\|z\|^2) \rangle \\ &+ \varphi'_5(z, \bar{z}). \end{aligned} \quad (3.24)$$

Similarly we can repeat the argument for the terms of degree 1 in u in (3.20) and separating types we obtain

$$\operatorname{Im} F_{3,1}^w(z, u) = 0, \quad (3.25)$$

$$u\|z\|^2 \operatorname{Re} F_{1,2}^w(z, 1) = u \operatorname{Re} \langle z, F_{2,1}^z(z, 1) \rangle + 2\|z\|^2 u \operatorname{Im} \langle z, F_{0,2}^z(1) \rangle. \quad (3.26)$$

Finally, repeating the process for the terms of degree 2 in u in (3.20), we obtain

$$\operatorname{Im} F_{1,2}^w(z, u) = 2 \operatorname{Re} \langle z, F_{0,2}^z(u) \rangle. \quad (3.27)$$

Now (3.22) and (3.25) imply that $F_{5,0}^w \equiv 0$ and $F_{3,1}^w \equiv 0$. Then (3.23) gives $F_{4,0}^z \equiv 0$. From (3.27), dividing both sides by u^2 and noticing that both maps $z \mapsto F_{1,2}^w(z, 1)$ and $z \mapsto \langle z, F_{0,2}^z(1) \rangle$ are holomorphic, we obtain

$$F_{1,2}^w(z, 1) \equiv 2i \langle z, F_{0,2}^z(1) \rangle. \quad (3.28)$$

Now substituting (3.28) into (3.24) (taking into account that $F_{1,2}^w(z, i\|z\|^2) = -\|z\|^4 F_{1,2}^w(z, 1)$ and $F_{0,2}^z(i\|z\|^2) = -\|z\|^4 F_{0,2}^z(1)$), we obtain

$$\varphi_5(z, \bar{z}) = \varphi'_5(z, \bar{z}) + 2\|z\|^2 \operatorname{Im}\langle z, F_{2,1}^z(z, 1) \rangle. \quad (3.29)$$

Also, from (3.26) and (3.28) we find

$$\operatorname{Re}\langle z, F_{2,1}^z(z, 1) \rangle = -4\|z\|^2 \operatorname{Im}\langle z, F_{0,2}^z(1) \rangle. \quad (3.30)$$

Separating types, this means

$$\langle F_{2,1}^z(z, 1), z \rangle = 4i\|z\|^2 \langle z, F_{0,2}^z(1) \rangle,$$

which, together with (3.29) implies

$$\varphi_5 - \varphi'_5 = -8\|z\|^4 \operatorname{Re}\langle z, F_{0,2}^z(1) \rangle. \quad (3.31)$$

Note that we have $\operatorname{tr}^2(\varphi_5) \equiv \operatorname{tr}^2(\varphi'_5) \equiv 0$. Therefore using the uniqueness of the trace decomposition (see [6]) we conclude that $\varphi_5 \equiv \varphi'_5$ and $F_{0,2}^z(1) = 0$, and hence $F_{1,2}^w(z, 1) \equiv 0$ in view of (3.28).

It remains to show that $\operatorname{Im} F_{0,3}^w(1) = 0$ and that the inequalities in (3.16) hold. We now can pass to the weighted order 6 inequality, which yields

$$\begin{aligned} & \varphi_6(z, \bar{z}, u) + \operatorname{Im} F_{6,0}^w(z) + \operatorname{Im} F_{4,1}^w(z, w) + \operatorname{Im} F_{2,2}^w(z, w) + \operatorname{Im} F_{0,3}^w(w) \\ & \geq 2 \operatorname{Re}\langle z, F_{5,0}^z(z) \rangle + 2 \operatorname{Re}\langle z, F_{3,1}^z(z, w) \rangle + 2 \operatorname{Re}\langle z, F_{1,2}^z(z, w) \rangle \\ & \quad + \|F_{1,1}^z(z, w)\|^2 + \varphi'_6(z, \bar{z}, u), \end{aligned} \quad (3.32)$$

where the terms are evaluated at $(z, w) = (z, u + i\|z\|^2)$. Using Corollary A.6 for the terms of degree 0 in (z, \bar{z}) , we have $u^3 \operatorname{Im} F_{0,3}^w(1) \geq 0$ which implies $\operatorname{Im} F_{0,3}^w(1) = 0$. Now we let $u = t\|z\|^2$ for $t \in \mathbb{R}$ and look at the weighted order 6 inequality using Lemma A.8 to pass to the terms of type (3, 3) in (z, \bar{z}) :

$$\begin{aligned} & \varphi_{3,3,0}(z, \bar{z}) + \varphi_{2,2,1}(z, \bar{z}, 1)t\|z\|^2 + \operatorname{Im}[F_{0,3}^w(1)(t+i)^3]\|z\|^6 \\ & \geq 2 \operatorname{Re}\langle z, F_{1,2}^z(z, 1)(t+i)^2\|z\|^4 \rangle + \|F_{1,1}^z(z, 1)(t+i)\|^2\|z\|^4 \\ & \quad + \varphi'_{3,3,0}(z, \bar{z}) + \varphi'_{2,2,1}(z, \bar{z}, 1)t\|z\|^2. \end{aligned} \quad (3.33)$$

In view of $\operatorname{Im} F_{0,3}^w = 0$, (3.33) leads to the quadratic inequality $Bt^2 + 2At + C \geq 0$ for all t and z with A, B, C as in (3.17). The latter inequality is clearly equivalent to (3.16). \square

Remark 3.4. Observe that, for any k , the property that one has the equality in (3.6) up to weighted order k does not depend on the choice of coordinates. Indeed, (3.6) is obtained by substituting the parametrization

$$\gamma: (z, u) \mapsto F\left(z, u + i\|z\|^2 + i \sum \varphi_\mu(z, \bar{z}, u)\right)$$

of $F(\partial D)$ into the defining function

$$\rho(z, w) := \operatorname{Im} w - \|z\|^2 - \sum \varphi'_\mu(z, \bar{z}, \operatorname{Re} w)$$

of $\partial D'$. Then the equality in (3.6) up to weighted order k means that $\rho \circ \gamma$ vanishes up to weighted order k at 0. Now we claim that for any smooth defining function $\tilde{\rho}$ of $\partial D'$ and any smooth parametrization $\tilde{\gamma}(\tilde{z}, \tilde{u}) = \gamma(z(\tilde{z}, \tilde{u}), u(\tilde{z}, \tilde{u}))$ of $F(\partial D)$ with $\frac{du}{d\tilde{z}}(0) = 0$, the weighted vanishing orders of $\tilde{\rho} \circ \tilde{\gamma}$ (in (\tilde{z}, \tilde{u})) coincide with that of $\rho \circ \gamma$ (in (z, u)). Indeed, we have $\tilde{\rho} = \rho\alpha$ for a suitable function α and hence the weighted vanishing order of $\tilde{\rho} \circ \tilde{\gamma}$ is at least as high as that of $\rho \circ \gamma$. Furthermore, writing $(z, u) = (A\tilde{z} + B\tilde{u}, C\tilde{u}) + O(\|\tilde{z}\|^2 + \tilde{u}^2)$ with suitable matrices A, B, C , we see that also the weighted vanishing order of $\tilde{\rho} \circ \tilde{\gamma}$ is at least as high as that of $\rho \circ \gamma$. Reversing the argument, we see that both vanishing orders are equal as claimed.

We shall say that $F(\partial D)$ is tangent to $\partial D'$ at 0 up to weighted order k if we have the equality in (3.6) up to weighted order k . The latter property is well defined and does not depend on coordinate choices in view of Remark 3.4.

Proposition 3.5. *Let $D, D' \subset \mathbb{C}^n$ be in their Chern–Moser normal forms and $F \in \mathcal{J}_{0,0}(D, D')$ be of the form (3.2). The following conditions are equivalent:*

- (1) *The germ F is 2-flat (in the sense of Definition 1.2);*
- (2) *$F(\partial D)$ is tangent to $\partial D'$ at 0 up to weighted order 4;*
- (3) *$\operatorname{Im}\langle z, F_{1,1}^z(z, 1) \rangle \equiv 0$.*

Proof. Suppose that F is 2-flat and choose coordinates according to Definition 1.2 such that $F = \operatorname{id} + O(3)$. By Theorem 3.1, $F_{3,0}^w \equiv 0$ and therefore we have the equality in (3.6) up to weighted order 3. Next, examining terms of weighted order 4 of types $(2, 2, 0)$ and $(0, 0, 2)$ in (z, \bar{z}, u) in (3.6) we see that they only involve the second derivatives of F and $\varphi_4^1 - \varphi_4^1$ (cf. (3.12)), where the latter vanishes by Lemma 3.3 since $F_{1,1}^z \equiv 0$. Hence, by Lemma A.8, the whole weighted homogeneous part of (3.6) of order 4 must vanish. Thus (1) implies (2).

Now assume (2). In particular, we have the equality in (3.11) which, for the terms of type $(0, 0, 2)$ in (z, \bar{z}, u) yields $\operatorname{Im} F_{0,2}^w(1) = 0$. Then the equality in (3.12) together with the trace decomposition implies (2) as in [6].

Finally, assuming (3), applying Lemma 3.3 and arguing as before, we obtain (2) proving that (2) and (3) are in fact equivalent. Now consider the parabolic automorphism of type

$$g_r(z, w) = \frac{(z, w)}{1 - rw}$$

with $r = -\operatorname{Re} F_{0,2}^w(1)$. As shown in [6], there exists a unique transformation h such that $h(0) = 0$, $d_0 h = \operatorname{id}$ and $\operatorname{Re} h_{0,2}^w(1) = 0$ and $\tilde{D}' := h(g_r(D'))$ is in its Chern–Moser normal form. Then the map $\tilde{F} = h \circ g_r \circ F$ satisfies (2) (with respect to D and \tilde{D}') and, moreover, $\operatorname{Re} \tilde{F}_{0,2}^w(1) = 0$. As we have seen, (2) implies (3) and therefore we can apply Lemma 3.3 (identity (3.15)) to \tilde{F} to conclude that $\tilde{F} = \operatorname{id} + O(3)$. Hence (1) holds as desired. \square

Proof of Theorem 1.4. By definition, if there exists a 2-flat map $F \in \mathcal{J}_{p,p'}(D, D')$, we have $F = \operatorname{id} + O(3)$ with respect to some Chern–Moser normal coordinates for ∂D and $\partial D'$ vanishing

at p and p' , respectively. Then by Lemma 3.3 (identity (3.13)), the Chern–Moser normal forms of ∂D and $\partial D'$ coincide up to weighted order 5 and therefore $(\partial D, p)$ and $(\partial D', p')$ are equivalent up to weighted order 5.

Conversely, suppose $(\partial D, p)$ and $(\partial D', p')$ are biholomorphically equivalent up to weighted order 5. Then it follows from the construction of the normal form in [6] that there exist Chern–Moser normal forms for ∂D and $\partial D'$ that coincide up to weighted order 5. We will construct a map $F \in \mathcal{J}(D, D')$ with $F = \text{id} + O(3)$ of the form

$$F(z, w) = (z, w) + (\lambda_1 w^2 z, \lambda_2 w^3 + i \lambda_3 w^4) \quad (3.34)$$

with $\lambda_1, \lambda_2, \lambda_3$ being real numbers to be suitably chosen. We first remark that with this choice of F one always has the equality in (3.6) up to weighted order 5. We now consider the corresponding inequality for the terms of weighted order 6,

$$\varphi_6^1(z, \bar{z}, u) + \lambda_2 \operatorname{Im}(u + i \|z\|^2)^3 \geq 2\lambda_1 \|z\|^2 \operatorname{Re}(u + i \|z\|^2)^2 + \varphi_6^2(z, \bar{z}, u), \quad (3.35)$$

which is equivalent to

$$u^2 \|z\|^2 (3\lambda_2 - 2\lambda_1) + \|z\|^6 (-\lambda_2 + 2\lambda_1) \geq \varphi_6^2(z, \bar{z}, u) - \varphi_6^1(z, \bar{z}, u), \quad (3.36)$$

where $\varphi_6^2(z, \bar{z}, u) - \varphi_6^1(z, \bar{z}, u) = O(\|z\|^6 + u^2 \|z\|^2)$. Therefore we can choose λ_1, λ_2 to have the strict inequality in (3.36) whenever $z \neq 0$. We still have the equality for $z = 0, u \neq 0$ and hence have to pass to higher order terms to obtain strict inequality for all $(z, u) \neq 0$. After further inspection of the terms of weighted orders 7 and 8 we see that each of them, except $\lambda_3 u^4$, is $o(\|z\|^6 + u^2 \|z\|^2)$ as $(z, u) \rightarrow 0$ due to the Chern–Moser normalization of the terms φ_μ^j . Hence, choosing $\lambda_3 > 0$ and λ_1, λ_2 as above we obtain the strict inequality for the sum of the terms up to weighted order 8 for all sufficiently small $(z, u) \neq 0$. Finally, in the full weighted homogeneous expansion of (3.6), we will also reach the strict inequality for all sufficiently small $(z, u) \neq 0$ implying $F \in \mathcal{J}_{p,p'}(D, D')$. This proves the existence part of Theorem 1.4. \square

Appendix A

A.1. Attraction property of analytic discs

The following elementary property has been used in the proof of Lemma 2.1 (see [4] for more elaborate refined versions).

Lemma A.1. *Let $D \subset \mathbb{C}^n$ be a bounded domain and $p \in \partial D$ a boundary point. Suppose that \bar{D} does not contain nontrivial complex-analytic varieties through p . Then, for any $0 < \eta < 1$ and any neighborhood U of p , there exists another neighborhood V of p such that, if $f: \Delta \rightarrow D$ is a holomorphic map with $f(0) \in V$, then $f(\eta\Delta) \subset U$.*

Proof. By contradiction, suppose that, for some fixed η and U , there exists a sequence of holomorphic maps $f_k: \Delta \rightarrow D$ with $f_k(0) \rightarrow p$ such that $f_k(\eta\Delta) \not\subset U$. By Montel's theorem, $\{f_k\}$ can be assumed convergent to a limit map $f: \Delta \rightarrow D$, uniformly on compacta, in particular, on $\eta\bar{\Delta}$. Since $f(\Delta) \subset \bar{D}$ and, by the assumption, \bar{D} does not contain nontrivial varieties through p , we must have $f(z) \equiv p$. The latter fact implies $f_k(\eta\Delta) \subset U$ contradicting the choice of the sequence $\{f_k\}$. The proof is complete. \square

A.2. Polynomial approximations in real variables

We begin with a function $f(x)$ in one (real) variable that is approximated by a polynomial $p(x)$ up to some error term $r(x)$. We have the following elementary property whose proof is left to the reader.

Lemma A.2. *Let $p(x)$ be a real polynomial of degree d , $r(x)$ a real function satisfying*

$$r(x) = o(|x|^d), \quad x \rightarrow 0,$$

and suppose that $p(x) + r(x) \geq 0$ for $x \geq 0$ in a neighborhood of 0. Then $p(x) \geq 0$ for $x > 0$ in a neighborhood of 0.

Remark A.3. The same statement obviously holds if $d > 0$ is replaced by any real number and $p(x)$ by any finite linear combination of powers x^l for $l \leq d$.

We next extend Lemma A.2 to several (real) variables. For simplicity, we restrict ourselves to the two-dimensional case. Recall that the *Newton polytope* of a polynomial $p(x_1, x_2) = \sum_{l_1 l_2} p_{l_1 l_2} x_1^{l_1} x_2^{l_2}$ is the convex hull of the set of all (l_1, l_2) with $p_{l_1 l_2} \neq 0$. The *extended Newton polytope* is the minimal convex set C containing the Newton polytope such that, if $(l_1, l_2) \in C$, then $(k_1, k_2) \in C$ whenever $k_1 \leq l_1$ and $k_2 \leq l_2$. We have the following extension of Lemma A.2.

Lemma A.4. *Let $p(x_1, x_2)$ be a real polynomial and for $j = 1, \dots, s$, let $r_j(x)$ be real functions and (d_{j1}, d_{j2}) be pairs of non-negative integers satisfying*

$$r_j(x) = o(|x_1^{d_{j1}} x_2^{d_{j2}}|), \quad x = (x_1, x_2) \rightarrow 0, \quad x_1, x_2 \geq 0.$$

Suppose that the convex hull of the set $\{(d_{j1}, d_{j2}) : 1 \leq j \leq s\}$ does not intersect the interior of the extended Newton polytope of $p(x)$ and that

$$p(x) + \sum_j r_j(x) \geq 0,$$

for $x_1, x_2 \geq 0$ in a neighborhood of 0. Then $p(x) \geq 0$ for $x_1, x_2 \geq 0$ in a neighborhood of 0.

Proof. It follows from the assumptions that there exists a pair $(v_1, v_2) \neq 0$ of non-negative integers such that, for any coefficient $p_{l_1 l_2} \neq 0$ of p and any $j = 1, \dots, s$, one has

$$v_1 l_1 + v_2 l_2 \leq v_1 d_{j1} + v_2 d_{j2}.$$

Then, for any real numbers $\lambda_1, \lambda_2 > 0$, we have $p(\lambda_1 x^{v_1}, \lambda_2 x^{v_2}) \geq 0$ for $x > 0$ in a neighborhood of 0 in view of Remark A.3. Since λ_1, λ_2 are arbitrary, we obtain the conclusion of the lemma. \square

Consider now the case of variables $X_1 \in \mathbb{R}^{n_1}$ and $X_2 \in \mathbb{R}^{n_2}$ and write a polynomial $p(X_1, X_2)$ in the form

$$p(X_1, X_2) = \sum_{l_1 l_2} p_{l_1 l_2}(X_1, X_2),$$

where $p_{l_1 l_2}(X_1, X_2)$ is bihomogeneous in (X_1, X_2) of bidegree (l_1, l_2) . Define the *extended bihomogeneous Newton polytope* of p in \mathbb{N}^2 in the same way as above. Then, we obtain the following extension of Lemma A.4.

Lemma A.5. *Let $p(X_1, X_2)$ be a real polynomial in $X = (X_1, X_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and for $j = 1, \dots, s$, let $r_j(x)$ be real functions and (d_{j1}, d_{j2}) pairs of non-negative integers satisfying*

$$r_j(X) = o(\|X_1\|^{d_{j1}} \|X_2\|^{d_{j2}}), \quad X = (X_1, X_2) \rightarrow 0.$$

Suppose that the convex hull of the set $\{(d_{j1}, d_{j2}) : 1 \leq j \leq s\}$ does not intersect the interior of the extended Newton polytope of $p(X)$ and that

$$p(X) + \sum_j r_j(X) \geq 0,$$

for X in a neighborhood of 0. Then $p(X) \geq 0$ for X in a neighborhood of 0.

The proof can be obtained by restricting p and r_j to the span of two arbitrary vectors $(v_1, 0)$ and $(0, v_2)$ in $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and applying Lemma A.4. In particular, we have the following “cancellation rule” for weighted homogeneous polynomials.

Corollary A.6. *Let $v_1, v_2 > 0$ be weights assigned to X_1, X_2 and let $p(X_1, X_2)$ be a weighted homogeneous polynomial of degree d in $(X_1, X_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, i.e. $P(t^{v_1} X_1, t^{v_2} X_2) = t^d P(X_1, X_2)$ and r be a real function satisfying*

$$r(X_1, X_2) = o((\|X_1\|^{1/v_1} + \|X_2\|^{1/v_2})^d), \quad (X_1, X_2) \rightarrow 0,$$

such that $p(X) + r(X) \geq 0$ for $X = (X_1, X_2)$ in a neighborhood of 0. Then $p(X) \geq 0$. Furthermore, if $p_0(X_1, X_2)$ is the nontrivial bihomogeneous component of p of minimal degree in X_1 (or in X_2), then also $p_0(X_1, X_2) \geq 0$.

A.3. Homogeneous polynomials in complex variables

By separating homogeneous terms and applying the above statements, one can reduce general polynomial inequalities to inequalities for homogeneous terms. We next state some elementary results that can be useful to separate complex monomials of the form $z^k \bar{z}^l$.

Let $p(z, \bar{z})$ be a homogeneous real-valued polynomial of degree d with

$$p(z, \bar{z}) = \sum_k p_k z^k \bar{z}^{d-k} \geq 0 \tag{A.1}$$

for $z \in \mathbb{C}$ in a neighborhood of 0.

Remark A.7. Observe that, if d is odd, then (A.1) is only possible if $p \equiv 0$. If d is even, the situation is more complicated. Set $d = 2s$. By integrating (A.1) for $z = z_0 e^{i\theta}$ with $0 \leq \theta \leq 2\pi$, we immediately obtain that $p_s \geq 0$.

In case $p_s > 0$ one has, in general, no conclusion about the other coefficients in (A.1). However, if $p_s = 0$, all other coefficients must vanish:

Lemma A.8. *Let $p(z, \bar{z})$ be a homogeneous real-valued polynomial of degree $2s$ satisfying (A.1) for $z \in \mathbb{C}$ in a neighborhood of 0. Suppose that $p_s = 0$. Then $p(z, \bar{z}) \equiv 0$.*

Proof. We assume $p \not\equiv 0$ and prove the statement by induction on the maximal number k with $p_k \neq 0$. By the assumption and the reality of p , we have $s < k \leq 2s$. Otherwise we have $s < k \leq 2s$ and let ε be any primitive $4(k-s)$ th root of unity. Then, if we multiply z in (A.1) by ε , we obtain a new inequality where the term with $z^k \bar{z}^{2s-k}$ changes sign whereas all other terms receive factors different from -1 . Hence, by adding the new inequality and the old one, we eliminate the term with $z^k \bar{z}^{2s-k}$ and keep all other nonzero terms with possibly changed but still nonzero coefficients. By the induction, the new polynomial must be zero. This is only possible if $z^k \bar{z}^{2s-k}$ and its conjugate are the only nonzero terms of $p(z, \bar{z})$. Since $k \neq s$, we obtain a contradiction with (A.1). Hence $p(z, \bar{z}) \equiv 0$. \square

References

- [1] M. Abate, Angular derivatives in strongly pseudoconvex domains, in: Proc. Sympos. Pure Math., vol. 52 (2), Amer. Math. Soc., Providence, RI, 1991, pp. 23–40.
- [2] M.S. Baouendi, P. Ebenfelt, L.P. Rothschild, Local geometric properties of real submanifolds in complex space, Bull. Amer. Math. Soc. (N.S.) 37 (3) (2000) 309–336.
- [3] L. Baracco, D. Zaitsev, G. Zampieri, A Burns–Krantz type theorem for domains with corners, preprint, <http://arxiv.org/abs/math.CV/0505264>.
- [4] F. Berteloot, Attraction des disques analytiques et continuité höldérienne d’applications holomorphes propres, in: Topics in Complex Analysis, Warsaw, 1992, Banach Center Publ., vol. 31, Polish Acad. Sci., Warsaw, 1995, pp. 91–98.
- [5] D.M. Burns, S.G. Krantz, Rigidity of holomorphic mappings and a new Schwarz lemma at the boundary, J. Amer. Math. Soc. 7 (3) (1994) 661–676.
- [6] S.S. Chern, J.K. Moser, Real hypersurfaces in complex manifolds, Acta Math. 133 (1974) 219–271.
- [7] C. Fefferman, The Bergman kernel and the holomorphic mappings of pseudoconvex domains, Invent. Math. 26 (1974) 1–65.
- [8] X. Huang, A boundary rigidity problem for holomorphic mappings on some weakly pseudoconvex domains, Canad. J. Math. 47 (2) (1995) 405–420.
- [9] H. Poincaré, Les fonctions analytiques de deux variables et la représentation conforme, Rend. Circ. Mat. Palermo 23 (1907) 185–220.
- [10] N. Tanaka, On the pseudo-conformal geometry of hypersurfaces of the space of n complex variables, J. Math. Soc. Japan 14 (1962) 397–429.